A recent investigation by Hottel et al.[5] indicates that the exterior heat loss also can be important in scaling of combustors. If this loss from the prototype is large, the additional similitude parameter discussed in Ref. 5 should be considered. Further problems are encountered if liquid fuel is used, but scaling of liquid fuel systems will not be considered here.

References

[1] Schaffer, A. and Cambel, A. B., "The effect of an opposing jet on flame stability," Jet Propulsion 25, 284–287 (1955).

[2] Way, S., "Similitude considerations in turbojet engines," Jet Propulsion 27, 162–168 (1957).

[3] Woodward, E. C., "Combustor similitude study," ARS J. 31, 1401–1407 (1961).

[4] Schaffer, A. and Cambel, A. B., "Continued investigations of the opposing jet flameholder," Jet Propulsion 26, 576–578 (1956).

[5] Hottel, H. C., Williams, G. C., Jensen, W. P., Tobey, A. C., and Burrage, P. M. R., "Modeling studies of baffle-type combustors," Ninth Symposium (International) on Combustion, Cornell University, Ithaca, N. Y., August 27–September 1, 1962.

Lagrangian and Hamiltonian Rocket Mechanics

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The equation of motion of a point-mass rocket in Lagrangian form valid in any coordinate system is derived. The Lagrangian form of the rocket equation can be used conveniently in the calculation of rocket motion involving complicated constraints for which the use of generalized dynamical coordinates is advantageous. As an example, a problem of rocket motion with moving constraints is discussed. The generalized Hamiltonian form of the rocket equation then is derived, and the same problem is worked out in Hamiltonian form.

Lagrangian

CONSIDERING the rocket as a point of variable mass moving with velocity $\dot{x}_i$ along any trajectory, the differential equations of rocket motion are

$$m\ddot{x}_i = \dot{m}c_i + F_i \qquad i = 1, 2, 3 \quad (1)$$

where

- t = time; dot above a letter specifies the differentiation with respect to time
- $m(t)$ = mass of the rocket at the instant t
- $\dot{m}$ = dm/dt = rate of fuel burning
- $x_i(t)$ = rectangular Cartesian coordinates of the rocket in fixed coordinate system at the instant t
- c_i = velocity of expelled mass relative to the rocket
- F_i = any applied force (excluding thrust force $\dot{m}c_i$); usually F_i is the sum of aerodynamic and gravity forces

Adding the term $\dot{m}\dot{x}_i$ to both sides of (1), one obtains

$$(d/dt)(m\dot{x}_i) = \dot{m}u_i + F_i \quad (2)$$

where $u_i = \dot{x}_i + c_i$ is the absolute velocity of the expelled mass in fixed coordinate system.

The generalized coordinates q_α are introduced by the set of equations

$$x_i = x_i(q_\alpha, t) \qquad i = 1, 2, 3 \qquad \alpha = 1, \ldots, n \quad (3)$$

where n is equal to 1, 2, or 3, depending upon the degrees of freedom of the rocket. If $n = 1$, the rocket is constrained to move along a curve, for $n = 2$ along the surface, and for $n = 3$ the rocket can move freely in space. Since, according to (3), x_i is not only a function of q_α but also of time t, the constraints can move in space.

The kinetic energy of the rocket at any instant t is given by

$$T = \tfrac{1}{2}m(t)\dot{x}_1\dot{x}_i \quad (4)$$

(Summation on repeated indices is implied.) In what follows, only indices i, j, α, and β will be used, whereby it is agreed that i and j will have always the values $i = 1, 2, 3$ and subscripts α and β have always the values $\alpha, \beta = 1, \ldots, n$, where n is the number of degrees of freedom of the rocket.

Dotting Eq. (2) by $\partial x_i/\partial q_\alpha$, one obtains

$$\frac{d}{dt}(m\dot{x}_i)\frac{\partial x_i}{\partial q_\alpha} = \dot{m}u_i\frac{\partial x_i}{\partial q_\alpha} + F_i\frac{\partial x_i}{\partial q_\alpha} \quad (5)$$

It can be verified readily that the left member of (5) can be expressed in the form

$$\frac{d}{dt}(m\dot{x}_i)\frac{\partial x_i}{\partial q_\alpha} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} \quad (6)$$

Hence, Eq. (5) can be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \dot{m}u_i\frac{\partial x_i}{\partial q_\alpha} + F_i\frac{\partial x_i}{\partial q_\alpha} \quad (7)$$

Equation (7) is the Lagrangian form of the rocket equation in generalized coordinates. The form (7) of the rocket equation can be used conveniently in any problem of rocket motion with moving constraints. The following example is chosen to demonstrate the solution of Eq. (7). A straight line describes a right circular cone above a vertical (x_3 axis) with a uniform angular speed ω. A small rocket of variable mass $m(t)$ moves along this rotating straight line without a friction. By integrating twice Eq. (7) the position of the rocket at any time t will be found.

Let θ be the angle that the straight line makes with the vertical. If x_i are rectangular coordinates of the rocket, then, according to Fig. 1,

$$\begin{aligned} x_1 &= r\sin\theta\cos\omega t \\ x_2 &= r\sin\theta\sin\omega t \\ x_3 &= r\cos\theta \end{aligned} \quad (8)$$

where r is the distance from the apex 0 to the rocket. Since

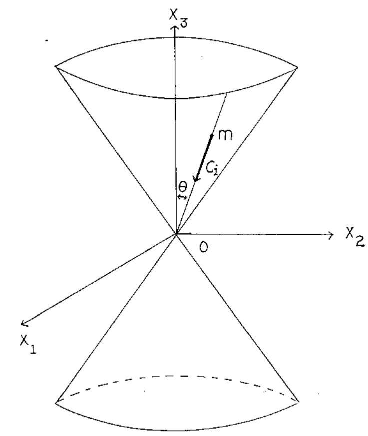

Fig. 1 Motion of rocket along a rotating straight line

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θ and ω are constant, $x_i = x_i(r, t)$, i.e., $q_1 = r$, $n = 1$, and (7) has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = \dot{m}(c_i + \dot{x}_i)\frac{\partial x_i}{\partial r} + F_i\frac{\partial x_i}{\partial r} \tag{9}$$

The rocket is subject to the uniform gravitational force mg_i oriented downward the x_3 axis, i.e., $g_i = (0, 0, -g)$, $F_i = (0, 0, -mg)$, where g = acceleration of gravity.

Suppose the relative velocity of expelled mass is c_i, where vector c_i is of constant magnitude c and is oriented along a rotating straight line toward the apex 0, i.e.,

$$c_i = -c(x_i/r) \tag{10}$$

The following relations can be verified easily:

$$T = \tfrac{1}{2}m\dot{x}_i\dot{x}_i = \tfrac{1}{2}m(\dot{r}^2 + \omega^2 r^2 \sin^2\theta) \tag{11}$$

The terms in the left member of (9) are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = \dot{m}\dot{r} + m\ddot{r} \qquad \frac{\partial T}{\partial r} = (m\omega^2 \sin^2\theta)r \tag{12}$$

The terms in the right member of (9) are

$$\dot{m}(c_j + \dot{x}_j)\frac{\partial x_j}{\partial r} = -\dot{m}c + \dot{m}\dot{r} \qquad F_j\frac{\partial x_j}{\partial r} = -mg\cos\theta \tag{13}$$

Substituting (11–13) into (9), one obtains the Lagrangian equation of motion:

$$m\ddot{r} - (m\omega^2\sin^2\theta)r = -\dot{m}c - mg\cos\theta \tag{14}$$

Suppose the rate of mass expulsion is given by

$$\dot{m} = -km \tag{15}$$

where k is constant.

In this case, the Lagrangian equation of motion reduces to

$$\ddot{r} - (\omega^2\sin^2\theta)r = kc - g\cos\theta \tag{16}$$

This is a linear equation with constant coefficients, whose solution with initial conditions

$$t = 0 \qquad r = r_0 \qquad \dot{r} = 0 \tag{17}$$

is given by

$$r = \left(r_0 + \frac{kc - g\cos\theta}{\omega^2\sin^2\theta}\right)\cosh\,[(\omega\sin\theta)t] + \frac{g\cos\theta - kc}{\omega^2\sin^2\theta} \tag{18}$$

Hamiltonian

If equations of transformation, Eqs. (3), contain time explicitly, the kinetic energy

$$\begin{aligned} T = \tfrac{1}{2}m\dot{x}_i\dot{x}_i &= \tfrac{1}{2}m\left(\frac{\partial x_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial x_i}{\partial t}\right)\left(\frac{\partial x_i}{\partial q_\beta}\dot{q}_\beta + \frac{\partial x_i}{\partial t}\right) \\ &= \tfrac{1}{2}m\frac{\partial x_i}{\partial q_\alpha}\frac{\partial x_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta + m\frac{\partial x_i}{\partial q_\alpha}\frac{\partial x_i}{\partial t}\dot{q}_\alpha + \frac{1}{2}m\frac{\partial x_i}{\partial t}\frac{\partial x_i}{\partial t} \end{aligned} \tag{19}$$

is, generally, a function of variables t, q_α, $\dot{q}_\alpha$, which will be written in abbreviated form as $T(t, q, \dot{q})$.

A new dynamic variable p_α is defined by the equation

$$p = \frac{T(t, q, \dot{q})}{\partial \dot{q}_\alpha} = m\frac{\partial x_i}{\partial q_\alpha}\frac{\partial x_i}{\partial q_\beta}\dot{q}_\beta + m\frac{\partial x_i}{\partial q_\alpha}\frac{\partial x_i}{\partial t} \tag{20}$$

It may be verified readily that, for the dynamical problems of constrained motion, the linear system (20) always can be solved for $\dot{q}_\alpha$ in terms of t, q_α, and p_α, i.e.,

$$\dot{q}_\alpha = \dot{q}_\alpha(t, q, p) \tag{21}$$

Introduce the function

$$h(t, q, \dot{q}) = \dot{q}_\alpha[\partial T(t, q, \dot{q})/\partial \dot{q}_\alpha] - T(t, q, \dot{q}) \tag{22}$$

or

$$h(t, q, \dot{q}) = \dot{q}_\alpha p_\alpha - T(t, q, \dot{q}) \tag{23}$$

The notation $h(t, q, \dot{q})$ and $T(t, q, \dot{q})$ specifies that the functions h and T are expressed in terms of variables t, q_α, $\dot{q}_\alpha$. Similarly, the notation $h(t, q, p)$ and $T(t, q, p)$ will indicate that the same functions h and T are expressed in terms of variables t, q_α, p_α, whereby the $\dot{q}_\alpha$ are expressed in terms of p_α by means of Eq. (20).

Take the first variation of (23); the usual symbol δ will be used for variation:

$$\begin{aligned} \delta h(t, q, \dot{q}) &= \dot{q}_\alpha\delta p_\alpha + p_\alpha\delta\dot{q}_\alpha - \left(\frac{\partial T}{\partial q_\alpha}\delta q_\alpha + \frac{\partial T}{\partial \dot{q}_\alpha}\delta\dot{q}_\alpha\right) \\ &= \dot{q}_\alpha\delta p_\alpha - \frac{\partial T}{\partial q_\alpha}\delta q_\alpha \end{aligned} \tag{24}$$

Taking variation of $h(t, q, p)$, one obtains

$$\delta h(t, q, p) = \frac{\partial h(t, q, p)}{\partial q_\alpha}\delta q_\alpha + \frac{\partial h(t, q, p)}{\partial p_\alpha}\delta p_\alpha \tag{25}$$

Since δq_α and δp_α are independent and $\delta h(t, q, p) \equiv \delta h(t, q, \dot{q})$ comparing (24) and (25) one obtains

$$\dot{q}_\alpha = \partial h(t, q, p)/\partial p_\alpha \tag{26}$$

and

$$-[\partial T(t, q, \dot{q})/\partial q_\alpha] = \partial h(t, q, p)/\partial q_\alpha \tag{27}$$

The Lagrange equation for rocket motion is given in Eq. (7)' which can be written in the following form:

$$\frac{\partial T(t, q, \dot{q})}{\partial q_\alpha} = \dot{p}_\alpha - \dot{m}u_i\frac{\partial x_i}{\partial q_\alpha} - F_i\frac{\partial x_i}{\partial q_\alpha} \tag{28}$$

Substituting (27) into (28), one obtains

$$\dot{p}_\alpha = -\frac{\partial h(t, q, p)}{\partial q_\alpha} + \dot{m}u_i\frac{\partial x_i}{\partial q_\alpha} + F_i\frac{\partial x_i}{\partial q_\alpha} \tag{29}$$

Equations (26) and (29) are the generalized Hamiltonian equations for the rocket mechanics.

If there exists a function $V(q, t)$ that can be called a generalized potential-energy function, such that

$$\dot{m}u_i\frac{\partial x_i}{\partial q_\alpha} + F_i\frac{\partial x_i}{\partial q_\alpha} = -\frac{\partial V(q, t)}{\partial q_\alpha} \tag{30}$$

then Eq. (26) can be written in the form

$$\dot{q}_\alpha = (\partial/\partial p_\alpha)[h(t, q, p) + V(q, t)] \tag{31}$$

since $\partial V(q, t)/\partial p_\alpha = 0$, and Eq. (29) in the form

$$\dot{p}_\alpha = -\partial/\partial q_\alpha)[h(t, q, p) + V(q, t)] \tag{32}$$

Defining the new function $H(t, q, p)$ by

$$H(t, q, p) = h(t, q, p) + V(q, t) \tag{33}$$

one obtains the canonical form of Hamilton's equations for the rocket mechanics:

$$\dot{q}_\alpha = \partial H(t, q, p)/\partial p_\alpha \tag{34}$$

$$\dot{p}_\alpha = -\partial H(t, q, p)/\partial q_\alpha \tag{35}$$

Now apply the Hamiltonian method to the example of rocket motion along a rotating straight line, which was solved by Lagrangian method:

$$T(t, q, \dot{q}) = \tfrac{1}{2}m(\dot{r}^2 + \omega^2 r^2\sin^2\theta)$$

$$p = \partial T(t, q, \dot{q})/\partial \dot{q} = \partial T(t, r, \dot{r})/\partial \dot{r} = m\dot{r}$$

$$\dot{r} = p/m$$

$$T(t, q, p) = \tfrac{1}{2}m(p^2/m^2 + \omega^2 r^2 \sin^2\theta)$$

$$h(t, q, p) = \tfrac{1}{2}(p^2/m) - \tfrac{1}{2}m\omega^2 r^2 \sin^2\theta$$

Hamilton's equations are

$$\dot{r} = p/m \tag{36}$$

$$\dot{p} = \frac{\partial h}{-\partial r} + (\dot{m}u_j + F_j)\frac{\partial x_j}{\partial r}$$

$$= (m\omega^2 \sin^2\theta)r - \dot{m}c + \dot{m}\dot{r} - mg\cos\theta \tag{37}$$

Eliminating $\dot{p}$ in (37) by means of (36), one obtains the differential equation

$$m\ddot{r} - (m\omega^2 \sin^2\theta)r = -\dot{m}c - mg\cos\theta$$

which was obtained previously by the Lagrangian method.

Interaction of Magnetohydrodynamic Simple Waves in Monatomic Fluids

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It is well known that the differential equations governing the one-dimensional nonsteady flow of an ideal, inviscid, compressible fluid may be reduced to a single second-order, linear, partial differential equation, the Euler-Poisson equation, whose general solution may be given explicitly in terms of two arbitrary functions for a set of values of the adiabatic index which includes those of usual physical interest.[1]

It is the purpose of this note to show that the interaction of hydromagnetic simple waves may be reduced to a single second-order, linear, partial differential equation, and the Riemann function thereof may be found from a Fourier superposition of the solutions of a linear, second-order, ordinary differential equation. Although it does not seem possible to solve this latter in terms of known functions, the advantage of the present method over other numerical methods, e.g., finite differences, is that the Riemann function is to be determined once and for all, and, thereafter, the solutions to specific initial-value problems may be given by quadratures.

The isentropic one-dimensional nonsteady motion of an ideal, inviscid, perfectly conducting monatomic compressible fluid subjected to a transverse magnetic field, i.e., the induction $\mathbf{B} = (0,0,B)$, is governed by the system of equations:

$$c_t + uc_x + (cu_x/3) = 0 \tag{1}$$

$$3cc_x + u_t + uu_x + b^2(B_x/B) = 0 \tag{2}$$

$$B_t + uB_x + Bu_x = 0 \tag{3}$$

where u, c, ρ, $b^2 = B^2/\mu\rho$, and μ are, respectively, the particle velocity, local speed of sound, density, square of the Alfvén speed, and permeability. Partial derivatives are denoted by subscripts, and all dependent variables are functions of x and t alone.

From the characteristic form of Eqs. (1–3), it has been shown[2, 3] that B/c^3 is constant along each particle path, and for a constant state or a simple-wave flow, B/c^3 was constant throughout the flow. This result still will obtain in the region of interaction of two simple waves, since the particle paths therein originate either in a constant state or a simple wave. Then if $B = r_1c^3$, $\rho = r_2c^3$, with constants r_1 and r_2, $b^2 = r_1^2c^3/\mu r_2 \equiv kc^3$, so that $\omega^2 = b^2 + c^2 = c^2(1 + kc)$, and the system of Eqs. (1–3) may be replaced by the system

$$x_\beta = (u + \omega)t_\beta \tag{4}$$

$$x_\alpha = (u - \omega)t_\alpha \tag{5}$$

$$u/2 + (1 + kc)^{3/2}/k = u/2 + (\omega/c)^3/k = \alpha \tag{6}$$

$$-u/2 + (1 + kc)^{3/2}/k = -u/2 + (\omega/c)^3/k = \beta \tag{7}$$

where (α, β) may be considered as generalizations of the usual Riemann invariants. In terms of (α,β), it was shown that

$$u = \alpha - \beta \tag{8}$$

$$kc = [k(\alpha + \beta)/2]^{2/3} - 1 \tag{9}$$

$$\omega = (\alpha + \beta)/2 - [k(\alpha + \beta)/2]^{1/3}/k \tag{10}$$

and from Eqs. (8–10) that an explicit solution for a centered simple wave could be given.[4]

The second-order, linear, partial differential equation is obtained by eliminating x from Eqs. (4) and (5) and using the relations given by Eqs. (8) and (10). This gives

$$t_{\alpha\beta} + \frac{(9\tau^{2/3} - 1)(t_\alpha + t_\beta)}{6(\alpha + \beta)(\tau^{2/3} - 1)} = 0 \tag{11}$$

where $\tau = k(\alpha + \beta)/2$. It is convenient to introduce the new dependent variable w by requiring that[5] $w_\alpha = x - (u + \omega)t$ and $-w_\beta = x - (u - \omega)t$, so that $w(\alpha, \beta)$ satisfies the equation

$$w_{\alpha\beta} + \frac{(3\tau^{2/3} + 1)(w_\alpha + w_\beta)}{6(\alpha + \beta)(\tau^{2/3} - 1)} = 0 \tag{12}$$

In order to transform Eq. (12) into a form for which results are known, a new dependent variable $v(\alpha, \beta)$ is introduced through the substitution $w = (\alpha + \beta)^{1/6}v(\alpha, \beta)/(\tau^{2/3} - 1)$. The resultant equation for v is

$$v_{\alpha\beta} + \left[\frac{9\tau^{2/3} + 7}{36(\alpha + \beta)^2(\tau^{2/3} - 1)}\right]v = 0 \tag{13}$$

The Riemann function of Eq. (13) may be obtained from the following result, which was derived heuristically by Riemann[5] and proved rigorously by Cohn.[6] Given the equation

$$v_{\alpha\beta} + H(\alpha + \beta)v = 0 \tag{14}$$

let $\xi = \alpha + \beta$, $\eta = \alpha - \beta$, and

$$R^* = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp[i\nu(\eta - \eta')]y(\xi, \xi'; \nu)d\nu$$

where $y(\xi,\xi'; \nu)$ is defined by the ordinary differential equation

$$(d^2y/d\xi^2) + [\nu^2 + H(\xi)]y = 0 \tag{15}$$

with $y = 0$, $dy/d\xi = 1$ for $\xi = \xi'$. Then, changing back to the variables (α, β), R^* satisfies the relation

$$R^* = \tfrac{1}{2}[\mathrm{sgn}(\alpha - \alpha') + \mathrm{sgn}(\beta - \beta')]R(\alpha, \beta;\ \alpha', \beta')$$

where $R(\alpha, \beta;\ \alpha', \beta')$ is the Riemann function for Eq. (14).

From this result, the Riemann function of Eq. (13) may be obtained from the Fourier superposition of the solutions of

$$\frac{d^2y}{d\xi^2} + \left[\nu^2 + \frac{9\sigma\xi^{2/3} + 7}{36\xi^2(\sigma\xi^{2/3} - 1)}\right]y = 0 \tag{16}$$

where $\sigma = (k/2)^{2/3}$, subject to the subsidiary conditions appended to Eq. (15). In order to put Eq. (16) into a form more convenient for comparison with known equations, let $\zeta = \sigma\xi^{2/3}$ and $y(\xi) = \zeta^{-1/4}s(\zeta)$. Then

$$\zeta(\zeta - 1)s'' + (1 - \zeta)s' + s[1 + \tfrac{9}{4}(\nu^2/\sigma^3)\zeta^2(\zeta - 1)] = 0 \tag{17}$$

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